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Individual aspects of quantum measurements

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Abstract. The following two claims have been put forward in support of an individual interpretation of quantum mechanics: within the framework of quantum measurement theory, (i) the concept of measurement of a sharp observable can be completely characterized in terms of the notion of calibration, and (ii) the frequency interpretation of probability (of a measurement outcome) can be founded on the notion of a property of an individual system. Here we set out to generalize these results to measurement schemes which may be unitary or non-unitary and to object and pointer observables which may be sharp or unsharp.

1. Introduction

Quantum mechanics has often been classified as a *merely* statistical ensemble theory, with not much bearings on the individual members of the ensembles. Yet there is an increasing variety of experiments exhibiting individual quantum processes which were conceived, devised and explained on the basis of this very theory. Therefore, in order to reach a proper appreciation of the scope of quantum mechanics, it is necessary to spell out the senses in which the theory does or does not apply to individual systems. In this paper we work out two special aspects of the quantum theory of measurement which are essential for an interpretation of quantum mechanics that refers to individual systems and their properties. We demonstrate the following:

(i) the theory of measurement for *sharp* observables can be based solely on the calibration condition;
(ii) a single measurement on a finite ensemble of identically prepared objects will most likely produce a sequence of outcomes with relative frequencies that are close to the quantum mechanical probabilities.

The first result gives a conceptually simple basis for the theory of measurements of sharp observables. The second result shows that the quantum mechanical probabilities emerge as definite properties—namely the relative frequencies of the pointer values—of an individual system consisting of a collection of measurement apparatus after a measurement. The ensemble is regarded here as an individual system, and the frequencies correspond to properties of that system.

The above results are already known to hold in various special cases. Here it is shown that they are valid in the most general context of quantum measurement theory, including unitary or non-unitary couplings, repeatable or non-ideal measurements, as well as sharp or unsharp object and pointer observables.

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2. Measurement scheme for an observable

Let E be an observable of the object system $\mathcal{S}$. Here we follow the formulation of quantum mechanics in which observables and states are represented as dual pairs of normalized positive operator valued measures and positive trace one operators acting on a complex separable Hilbert space $\mathcal{H}$. Let Ω be a set and $\mathcal{F}$ a σ-algebra of subsets of Ω, and let $\mathcal{L}(\mathcal{H})$ denote the set of bounded oparators on $\mathcal{H}$. We recall that $E : \mathcal{F} \to \mathcal{L}(\mathcal{H})$ is a normalized positive operator valued measure if (i) $E(\Omega) = I$ (normalization), (ii) $E(X) \geqslant O$ (positivity), and (iii) $E(\cup X_i) = \sum E(X_i)$ for all disjoint sequences $(X_i) \subset \mathcal{F}$ (σ-additivity) (where the sum converges in the weak operator topology of $\mathcal{L}(\mathcal{H})$). If, in addition, $E(X)^2 = E(X)$ for all $X \in \mathcal{F}$, then E is a projection operator valued measure. An observable is a *sharp observable* if it is represented by a projection operator valued measure. A particular case of sharp observables are those given by self-adjoint operators which are in one-to-one onto correspondence with the projection operator valued measures defined on the Borel subsets of the real line $\mathbb{R}$. A state T is a positive trace one operator on $\mathcal{H}$. A particular class of states are the *vector states* (one-dimensional projection operators) $T = P[\varphi]$ generated by the unit vectors $\varphi \in \mathcal{H}$; $P[\varphi]\psi := \langle \varphi \mid \psi \rangle \varphi$, $\psi \in \mathcal{H}$. Any observable E and state T determines a probability measure $p^E_T : X \mapsto p^E_T(X) := \mathrm{tr}\, T E(X)$ for which the minimal interpretation is adopted: the number $p^E_T(X)$ is the probability that a measurement of the observable E on the system in the state T leads to a result in the set X. Further details of this formulation of quantum mechanics are given, for instance, in [**?**]. E assigns a probability measure $p^E_T : X \mapsto p^E_T(X) := \mathrm{tr}\, T E(X)$ to any state T of the system. On the minimal interpretation, the number $p^E_T(X)$ is the probability that a measurement of E on $\mathcal{S}$ in the state T leads to a result (in the set) X. The intuitive starting point of the theory of measurement of an observable is the identification of the assumption that a measurement leads to a result with the assumption that the pointer observable has a corresponding value after the measurement. In that view the probability $p^E_T(X)$ should equal the probability that the pointer observable will have a corresponding value after the measurement. However, apart from the trivial case of $p^E_T(X)$ being one (or zero) it is a most delicate issue in quantum mechanics, known as the measurement problem, to explain that an observable, such as the pointer, does have a value with some probability†. Therefore, it would be desirable to formulate the concept of measurement without any reference to this problematic requirement. This can be achieved for an important class of observables, the sharp ones. To show this we introduce first the notion of a measurement scheme. For further motivation and details of the formulation of a quantum measurement process the reader may wish to consult [**?**].

A *measurement scheme* $\mathcal{M} := \langle \mathcal{H}_\mathcal{A}, T_\mathcal{A}, P_\mathcal{A}, V \rangle$ for the object system $\mathcal{S}$ consists of

- a measuring apparatus $\mathcal{A}$, with a Hilbert space $\mathcal{H}_\mathcal{A}$,
- an initial state $T_\mathcal{A}$ of $\mathcal{A}$,
- a pointer observable $P_\mathcal{A}$ of $\mathcal{A}$, and
- a measurement coupling V, given as a linear state transformation of the object–apparatus system $\mathcal{S} + \mathcal{A}$.

Formally, a measurement scheme $\mathcal{M}$ determines an observable $E_\mathcal{M}$ of the object system $\mathcal{S}$ such that the measurement outcome probabilities for this observable are recovered from those of the pointer observable after the measurement: for any (initial) state T of $\mathcal{S}$ and for any set X (of values of the pointer)

$$p^{E_\mathcal{M}}_T(X) = p^{P_\mathcal{A}}_{\mathcal{R}_\mathcal{A}(V(T\otimes T_\mathcal{A}))}(X) \tag{1}$$

† We take $p^E_T(X) = 1$ to ensure that E has a *definite* value X in the state T.

where $\mathcal{R}_{\mathcal{A}}(V(T\otimes T_{\mathcal{A}}))$ is the (reduced) state of the apparatus after the measurement, obtained by tracing out the object degrees of freedom from the final object–apparatus state $V(T\otimes T_{\mathcal{A}})$. We then say that a measurement scheme $\mathcal{M}$ for $\mathcal{S}$ is a *measurement* of an observable E if $E_{\mathcal{M}} = E$, that is, if $\mathcal{M}$ satisfies the *probability reproducibility condition* with respect to E:

$$p_T^E(X) = p^{P_{\mathcal{A}}}_{\mathcal{R}_{\mathcal{A}}(V(T\otimes T_{\mathcal{A}}))}(X) \tag{2}$$

for all T, X. This condition reflects the idea that a measurement should reproduce the outcome probabilities for the measured observable in the final distribution of the pointer values. It turns out that in the context of measurements of sharp observables the probability reproducibility condition is a consequence of an apparently weaker condition that can be imposed on a measurement scheme $\mathcal{M}$—the calibration condition—which *prima facie* is free of problematic interpretational implications.

3. Calibration condition and probability reproducibility

A measurement scheme $\mathcal{M}$ for $\mathcal{S}$ is said to satisfy the *calibration condition* with respect to an observable E if for any T and X the following implication holds true:

$$\text{if } p_T^E(X) = 1 \Longrightarrow \text{ then } p^{P_{\mathcal{A}}}_{\mathcal{R}_{\mathcal{A}}(V(T\otimes T_{\mathcal{A}}))}(X) = 1. \tag{3}$$

Clearly, if $\mathcal{M}$ satisfies the probability reproducibility condition (2) with respect to E, then it also satisfies the calibration condition (3) with respect to this observable. Our first result shows that the converse is also true whenever E is a sharp observable.

Theorem 1. A measurement scheme $\mathcal{M}$ for $\mathcal{S}$ satisfies the calibration condition with respect to a sharp observable E if and only if it satisfies the probability reproducibility condition with respect to this observable.

Proof. Assume that the measurement scheme $\mathcal{M}$ fulfills the calibration condition with respect to a sharp observable E. Consider first a vector state $T = P[\varphi]$ of $\mathcal{S}$. If $E(X)P[\varphi] = P[\varphi]$, then, by (3), equation (2) holds. On the other hand, if $E(X)P[\varphi] = O$, then, by (3), one also has $p^{P_{\mathcal{A}}}_{\mathcal{R}_{\mathcal{A}}(V(P[\varphi]\otimes T_{\mathcal{A}}))}(X) = 0$. Assume next that $P[\varphi]$ is such that $0 \neq p^E_{P[\varphi]}(X) \neq 1$. Using the identity

$$P[\varphi] = E(X)P[\varphi]E(X) + E(X')P[\varphi]E(X') + E(X)P[\varphi]E(X') + E(X')P[\varphi]E(X)$$

and applying the calibration condition (3), one gets

$$p^{I\otimes P_{\mathcal{A}}}_{V(P[\varphi]\otimes T_{\mathcal{A}})}(X) = p^E_{P[\varphi]}(X) + 2\mathrm{Re}\left\{\mathrm{tr}\, I\otimes P_{\mathcal{A}}(X)V(E(X)P[\varphi]E(X')\otimes T_{\mathcal{A}})\right\}.$$

We show that the last term vanishes identically, so that the probability reproducibility condition is obtained. To this end we write $T_X := p^E_{P[\varphi]}(X)^{-1}E(X)P[\varphi]E(X)$ and observe that

$$1 = \mathrm{tr}\, I\otimes P_{\mathcal{A}}(X)V(T_X\otimes T_{\mathcal{A}}) = \mathrm{tr}\, V^*\big(I\otimes P_{\mathcal{A}}(X)\big)\, T_X\otimes T_{\mathcal{A}}$$

where we have introduced the dual map V^* of V (see, e.g., [**?**]). Let $T_{\mathcal{A}} = \sum w_i P[\phi_i]$ be a vector state decomposition of $T_{\mathcal{A}}$. It follows that for any $P[\phi_i]$ one also has

$$\mathrm{tr}\, V^*\big(I\otimes P_{\mathcal{A}}(X)\big)\, T_X\otimes P[\phi_i] = 1$$

and therefore

$$V^*\big(I \otimes P_{\mathcal{A}}(X)\big)\, E(X) \otimes P[\phi_i](\varphi \otimes \psi) = E(X) \otimes P[\phi_i](\varphi \otimes \psi)$$

for all $\varphi \in \mathcal{H}_{\mathcal{S}}$, $\psi \in \mathcal{H}_{\mathcal{A}}$. Thus

$$V^*\big(I \otimes P_{\mathcal{A}}(X)\big)\, E(X) \otimes P[\phi_i] = E(X) \otimes P[\phi_i]$$

and

$$E(X') \otimes P[\phi_i]\, V^*\big(I \otimes P_{\mathcal{A}}(X)\big)\, E(X) \otimes P[\phi_i] \;=\; 0.$$

Taking the expectation with respect to $\varphi \otimes \phi_i$ yields

$$\mathrm{tr}\, V^*\big(I \otimes P_{\mathcal{A}}(X)\big)\, E(X) P[\varphi] E(X') \otimes P[\phi_i] = 0.$$

But this gives

$$\begin{aligned}
&\mathrm{tr}\, I \otimes P_{\mathcal{A}}(X) V(E(X)P[\varphi]E(X') \otimes T_{\mathcal{A}}) \\
&\quad = \mathrm{tr}\, V^*\big(I \otimes P_{\mathcal{A}}(X)\big)\, E(X) P[\varphi] E(X') \otimes T_{\mathcal{A}} \\
&\quad = \sum w_i\, \mathrm{tr}\, V^*\big(I \otimes P_{\mathcal{A}}(X)\big)\, E(X) P[\varphi] E(X') \otimes P[\phi_i] = 0.
\end{aligned}$$

If $T = \sum t_i P[\psi_i]$, then one has $p^E_T(X) = \sum t_i p^E_{P[\psi_i]}(X) = \sum t_i p^{P_{\mathcal{A}}}_{\mathcal{R}_{\mathcal{A}}(V(P[\psi_i]\otimes T_{\mathcal{A}}))}(X) = p^{P_{\mathcal{A}}}_{\mathcal{R}_{\mathcal{A}}(V(T\otimes T_{\mathcal{A}}))}(X)$, which completes the proof. □

This result had already been obtained in [**?**] for a special class of unitary measurements of discrete sharp observables, that is, for the measurement schemes in which the coupling V is effected by a unitary map U on $\mathcal{H}_{\mathcal{S}} \otimes \mathcal{H}_{\mathcal{A}}$, the apparatus $\mathcal{A}$ is prepared in a vector state $P[\phi]$, the pointer observable $P_{\mathcal{A}}$ is sharp, and the measured observable E is both sharp and discrete. The generalization to arbitrary sharp observables and arbitrary initial states of $\mathcal{A}$ was achieved by Herbut [**?**]. The present proof is more straightforward and exhaustive, encompassing non-unitary measurements and possibly unsharp pointers as well. This is physically desirable as the $\mathcal{S} + \mathcal{A}$ dynamics will be non-unitary in realistic situations due to interactions of the macroscopic parts of $\mathcal{A}$ with its environment; also, it can be argued that macroscopic pointer observables are essentially unsharp observables [**?**].

Theorem 1 shows that for sharp observables the theory of measurement can be based on the simple notion of calibration: a measurement exhibits unequivocally what is the case; that is, if the measured observable has a particular value before the measurement (in the sense that $p^E_T(X) = 1$), then the pointer observable has the corresponding value after the measurement (in the sense that $p^{P_{\mathcal{A}}}_{\mathcal{R}_{\mathcal{A}}(V(T\otimes T_{\mathcal{A}}))}(X) = 1$). It may be helpful to note that with this formulation we do not stipulate that an observable has a definite value *only* when the system is in an eigenstate of that observable; but we do regard the latter condition as sufficient for the former, cp the footnote on page 2. In this way the concept of measurement can, for sharp observables, be reduced to the concept of calibration within the frame of measurement theory. Furthermore, theorem 1 underlines the fundamental nature and inevitability of the probability reproducibility condition, which was originally taken as the defining criterion for the term *measurement*. This provides strong support for invoking this condition also in the definition of measurements of unsharp observables (for which the calibration condition, in general, is not sufficient for the whole probability reproducibility condition). Finally, theorem 1 sharpens the measurement problem, that is, the problem of justifying the interpretation of the measurement outcome probabilities for the measured observable as the actual *distribution* of the pointer values after the measurement. This problem is now seen to arise already from the calibration condition in conjunction with the assumption of a unitary measurement coupling [**?**]. In particular, the repeatability property of a measurement, which is usually assumed in this context, is *not* essential to establishing the measurement problem.

4. Emergence of outcome probabilities as relative frequencies

Next we work out an ensemble interpretation of the quantum mechanical probabilities for arbitrary observables, whether sharp or unsharp. In formulating this interpretation, we consider n runs of the same measurement, performed on n identically prepared copies of the object system $\mathcal{S}$, as *one single physical process* to be described by quantum mechanics. Regarding this theory as universally valid and complete, one would expect it to be able to predict that in a large system consisting of n equally prepared systems $\mathcal{S}$ the relative frequency of any outcome after a measurement would be almost equal to the corresponding quantum mechanical probability. This expectation will be confirmed in the form of theorem 2. Technically we make use of similar theorems proved earlier by various authors in the context of the many-worlds interpretation of quantum mechanics. Our approach has various advantages. First, we admit arbitrary object observables and do not restrict ourselves to repeatable measurements; hence, for example, genuinely unsharp measurements which allow no reduction to, or approximation by, sharp measurements, are taken into account. A realistic example of such genuinely unsharp measurements is furnished by quantum optical schemes affording joint measurement of two conjugate quadrature components—hence phase space measurements. See, e.g., [**?**]. Further examples and discussions can be found in [**?**]. Second, neither do we assume the pointer to be a sharp observable; finally, we will allow for arbitrary (linear, but not necessarily unitary) measurement couplings.

Let $\mathcal{S}^{(n)}$ be an n-body system consisting of n identical copies of $\mathcal{S}$: $\mathcal{S}^{(n)} = \mathcal{S}_1 + \ldots + \mathcal{S}_n$. The associated Hilbert space is the tensor product Hilbert space $\mathcal{H}^{(n)} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$. A measurement scheme $\mathcal{M}$ for $\mathcal{S}$ (= $\mathcal{S}_1 = \cdots = \mathcal{S}_n$) can be extended to a measurement scheme $\mathcal{M}^{(n)}$ for $\mathcal{S}^{(n)}$ by forming the n-fold tensor products of the constituents of $\mathcal{M}$. In order to collect the statistics, one needs to fix a reading scale $\mathcal{R} = (X_i)_{i\in\mathbb{I}}$. A *reading scale* is a countable partition $(X_i)_{i\in\mathbb{I}} \subset \mathcal{F}$ of the value space Ω of the pointer observable. It serves to discretize the readout space, discretizing in the first instance the pointer observable and thus also the measured observable. The reading scale $\mathcal{R} = (X_i)_{i\in\mathbb{I}}$ gives rise to a discretized pointer observable $P_{\mathcal{A}}^{\mathcal{R}} : i \mapsto P_i := P_{\mathcal{A}}(X_i)$, $X_i \in \mathcal{R}$, and thus to a discretized measurement scheme $\mathcal{M}^{\mathcal{R},(n)}$ for $\mathcal{S}^{(n)}$. A typical measurement outcome sequence of this scheme is $\ell \equiv (l_1, \ldots, l_n)$, with $l_k \in \mathbb{I}$, $k = 1, \ldots, n$. For any $X_i \in \mathcal{R}$ with $p_T^E(X_i) \neq 0$, the *final component state* of $\mathcal{A}$ conditional on the occurrence of an outcome X_i is

$$p_T^E(X_i)^{-1} P_i^{1/2} \mathcal{R}_{\mathcal{A}}(V(T \otimes T_{\mathcal{A}})) P_i^{1/2} =: T_{\mathcal{A}}(i, T). \tag{5}$$

(For $p_T^E(X_i) = 0$, we put $T_{\mathcal{A}}(i, T) = O$). Since the pointer observable $P_{\mathcal{A}}$ is not assumed to be a sharp observable one has $O \leqslant P_i^2 \leqslant P_i \leqslant P_i^{1/2} \leqslant I$ and thus only the inequality

$$\operatorname{tr} T_{\mathcal{A}}(i, T) P_i = p_T^E(X_i)^{-1} \operatorname{tr} \mathcal{R}_{\mathcal{A}}(V(T \otimes T_{\mathcal{A}})) P_i^2 \leqslant 1. \tag{6}$$

We assume from now on that the measurement scheme $\mathcal{M}$ fulfills the *pointer value-definiteness condition* with respect to the reading scale $\mathcal{R}$ (a fundamental theorem of the quantum theory of measurement [**?**] assures that for each observable E there are unitary measurements with sharp pointer observables; for such measurements the pointer value-definiteness condition is automatically fulfilled (for a review, see the revised edition of [**?**])): for each $X_i \in \mathcal{R}$ and any T with $p_T^E(X_i) \neq 0$

$$\operatorname{tr} T_{\mathcal{A}}(i, T) P_i = 1. \tag{7}$$

In order that the states $T_{\mathcal{A}}(i, T)$ can be interpreted as the conditional final states of $\mathcal{A}$, it is necessary that the reduced state $\mathcal{R}_{\mathcal{A}}(V(T \otimes T_{\mathcal{A}})) \equiv T_{\mathcal{A}}(\Omega, T)$ can be expressed as a mixture of these states; that is, one must stipulate the *pointer mixture condition* for $\mathcal{A}$:

$$T_{\mathcal{A}}(\Omega, T) = \sum \mathrm{tr}\, T_{\mathcal{A}}(\Omega, T) P_i T_{\mathcal{A}}(i, T). \tag{8}$$

Condition (7) entails, in particular, that the operators P_i have the eigenvalue 1 (though they need not be projections). We denote the spectral projection of P_i associated with this eigenvalue as $P_i^{(1)}$. Let $P_\ell^{(n)} := P_{l_1}^{(1)} \otimes \cdots \otimes P_{l_n}^{(1)}$, for all ℓ. For each $i \in \mathbb{I}$ one may define a *relative frequency operator* [**?**]

$$F_i^{(n)} = \sum_\ell f_i^{(n)}(\ell) P_\ell^{(n)} \tag{9}$$

with the eigenvalues

$$f_i^{(n)}(\ell) = \frac{1}{n} \sum_{j=1}^{n} \delta_{l_j, i}. \tag{10}$$

Let $T_{\mathcal{A}}^{(n)}(\ell, T) := T_{\mathcal{A}}(l_1, T) \otimes \cdots \otimes T_{\mathcal{A}}(l_n, T)$ be the final component state of the apparatus $\mathcal{A}^{(n)}$. The eigenvalue equation

$$F_i^{(n)} T_{\mathcal{A}}^{(n)}(\ell, T) = f_i^{(n)}(\ell) T_{\mathcal{A}}^{(n)}(\ell, T) \tag{11}$$

shows that the relative frequency of the pointer value i (or X_i) corresponds to a real property in the final component state of the apparatus, a property which is given by the eigenvalue $f_i^{(n)}(\ell)$ of the relative frequency operator $F_i^{(n)}$. In accordance with the assumption that an observable *has* a particular value in a given state if this state assigns probability one to that value (see the footnote on page 2), we say that a property is *real* in a state if this state assigns probability one to this property.

The *statistical ensemble interpretation* of probability (see, e.g., [**?**]), which is at issue here, states that if one performed a large number of E-measurements, with a fixed reading scale $\mathcal{R}$, on systems $\mathcal{S}$ equally prepared in state T, then the relative frequency of the outcomes i, would approach the probability $p_T^E(X_i)$. Hence probability is related again to the situation after the measuring process, that is, to the final state of $\mathcal{A}^{(n)}$:

$$\begin{aligned} T_{\mathcal{A}}^{(n)}(\Omega, T) &:= \mathcal{R}_{\mathcal{A}}\big(V(T \otimes T_{\mathcal{A}})\big) \otimes \cdots \otimes \mathcal{R}_{\mathcal{A}}\big(V(T \otimes T_{\mathcal{A}})\big) \\ &= \sum_\ell p_\ell^{(n)}\, T_{\mathcal{A}}^{(n)}(\ell, T) \end{aligned} \tag{12}$$

where the last equality is a consequence of the pointer mixture condition (8) and $p_\ell^{(n)} = p_{l_1} \cdot \cdots \cdot p_{l_n}$, with $p_i = \mathrm{tr}\, T_{\mathcal{A}}(\Omega, T) P_i$.

The expectation and variance of the frequency operator $F_i^{(n)}$ in the state $T_{\mathcal{A}}^{(n)}(\Omega, T)$ are found to be (the proof of these equations are in complete analogy to those of Hartle [**?**], as well as those in [**?**, **?**]; here we follow the adaptation worked out in [**?**])

$$\mathrm{Exp}\big(F_i^{(n)}; T_{\mathcal{A}}^{(n)}(\Omega, T)\big) = \mathrm{tr}\left[T_{\mathcal{A}}^{(n)}(\Omega, T)\, F_i^{(n)}\right] = p_i \tag{13a}$$

$$\begin{aligned} \mathrm{Var}\big(F_i^{(n)}; T_{\mathcal{A}}^{(n)}(\Omega, T)\big) &= \mathrm{tr}\left[T_{\mathcal{A}}^{(n)}(\Omega, T)\left(F_i^{(n)} - p_i\right)^2\right] \\ &= \sum_\ell \left(f_i^{(n)}(\ell) - p_i\right)^2\, p_\ell^{(n)} = \frac{1}{n} p_i\,(1 - p_i)\,. \end{aligned} \tag{13b}$$

The expressions on the right-hand sides are verified by induction with respect to n. The probabilities p_i are thus recovered, in the limit of large n, as relative frequencies of the pointer values i. In other words, the uncertainty about the pointer value of the individual apparatus system, when viewed from the point of view of the individual system, in the state $T_{\mathcal{A}}^{(n)}(\Omega, T)$ of (13), consisting of many apparatus, is turned into a fairly 'sharp' distribution, with weights $p_\ell^{(n)}$, of sequences of frequencies of pointer values. This result can be rephrased as a statement about a measurement performed on the object system as follows. Let E be the observable determined by the measurement scheme $\mathcal{M}$. The defining condition (1) implies that $p_i = p_T^E(X_i)$. Similarly, the scheme $\mathcal{M}^{(n)}$ qualifies as a measurement of the observable $E^{(n)} = E \otimes \cdots \otimes E$ of an ensemble $\mathcal{S}^{(n)}$ of n systems $\mathcal{S}$ in the state $T^{(n)} = T \otimes \cdots \otimes T$. Hence the object probabilities $p_T^E(X_i)$ are themselves tied equally well to the frequency values of the ensemble of apparatus.

Theorem 2. Let $\mathcal{M}$ be a measurement of an observable E such that the pointer value-definiteness condition (7) is fulfilled with respect to a reading scale $\mathcal{R}$. For any state T, and all $X_i \in \mathcal{R}$, one has

$$\mathrm{Exp}\big(F_i^{(n)}; T_{\mathcal{A}}^{(n)}(\Omega, T)\big) = p_T^E(X_i) \tag{14a}$$

$$\lim_{n\to\infty} \mathrm{Var}\big(F_i^{(n)}; T_{\mathcal{A}}^{(n)}(\Omega, T)\big) = 0. \tag{14b}$$

According to this theorem the relative frequency of the pointer value i after a measurement $\mathcal{M}^{\mathcal{R},(n)}$ on an ensemble of n systems $\mathcal{S}$ in the state $T^{(n)}$ approaches the probability $p_T^E(X_i)$ in the limit of large n.

It is important to recall that the approximation of the object probabilities p_i by relative frequencies is itself a probabilistic statement involving probabilities about the large ensemble of measuring apparatus. This is a reflection of the fact that the concept of probability cannot be reduced to that of relative frequency. But what is essential to the individual interpretation of quantum mechanics is the fact that on the ensemble level, viewed as an individual system, one obtains statements involving probabilities close to unity, so that the corresponding properties can be asserted almost with certainty. To see this more clearly, let us turn the limit statement (14*b*) into the form known as Bernoulli's theorem (see, e.g., [**?**]). We note first that the eigenvalues $f_i^{(n)}(\ell)$ are degenerate since according to (10), for any permutation $\pi(\ell)$ of a sequence ℓ one has $f_i^{(n)}\big(\pi(\ell)\big) = f_i^{(n)}(\ell)$. Therefore the spectral projection of $F_i^{(n)}$ associated with an eigenvalue $f_i^{(n)}(\ell)$ is $\sum_\pi P_{\pi(\ell)}^{(n)} =: \Pi_{[\ell]}^{(n)}$. Here the summation runs over all permutations π which do not permute identical elements of ℓ among themselves; and $[\ell]$ denotes the class of all sequences resulting from such permutations of a sequence ℓ. For a positive number ϵ we define

$$\mathcal{P}_\epsilon^{(n)} := \sum_{[\ell]:|f_i^{(n)}(\ell)-p_i|\leqslant\epsilon} \Pi_{[\ell]}^{(n)}. \tag{15}$$

Condition (14*b*) then implies that

$$\lim_{n\to\infty} \mathrm{tr}\, T_{\mathcal{A}}^{(n)}(\Omega, T)\, \mathcal{P}_\epsilon^{(n)} = 1. \tag{16}$$

This means that for any positive ϵ the probability for the frequency being close within ϵ to the intended probability p_i approaches one as $n \to \infty$. In this sense the frequency, which is closest to p_i, is the one that will most likely have been realized in the distribution of pointer positions of the ensemble at the end of the measurement. One may wonder whether a more specific statement could be achieved, in the sense that for some of the spectral projections $\Pi^{(n)}_{[\ell]}$ the expectation value would approach unity. However, one can show (for the case of the outcome label set $\mathbb{I}$ being finite) that $\max_\ell \left\{ \mathrm{tr}\, T^{(n)}_{\mathcal{A}}(\Omega, T)\, \Pi^{(n)}_{[\ell]} \right\} \to 0$ as $n \to \infty$ (as long as all $p_k \neq 1$). Hence it cannot be maintained that in the *finite* ensembles the probabilities p_k would be approached by the corresponding frequencies in this somewhat stronger sense. On the other hand, it has been shown recently that in the context of *infinite* ensembles the probability may assume the status of a definite property. Indeed, Coleman and Lesniewski [**?**] have constructed a 'randomness operator', based on the classical notion of the randomness of a sequence of the numbers ± 1. This operator projects onto the subspace of eigenstates of the frequency operator for a spin-$\frac{1}{2}$ observable s_x, say, associated with *random* sequences ℓ. Then they showed that the infinite products of s_z eigenstates belong to that subspace. Using that result, Gutman [**?**] has constructed projections for an infinite ensemble of spin-$\frac{1}{2}$ systems which represent probabilities for the individual members as real properties in the sense of some frequencies for the ensemble. In this way the recovery of probabilities from properties can be achieved without any problematic limiting procedures if the notion of infinite ensembles is accepted.

5. Conclusion

In summary, we have presented two theorems underlining the possibilites of presenting a foundation of quantum mechanics on concepts pertaining to individual systems rather than statistical concepts. Theorem 1 ensures that in the case of sharp observables the concept of measurement can be based solely on the calibration requirement; the linearity of the measurement dynamics then entails the full probability reproducibility requirement. The latter condition is therefore rightfully adopted as the appropriate generalization fixing the term measurement for sharp as well as unsharp observables. This is very satisfying, taking into account that for unsharp observables the calibration condition is not applicable.

Theorem 2 shows in which sense a measurement outcome probability emerges as a property—the value of a frequency observable—of an ensemble of measuring apparatus viewed as an individual system. Our result provides a thorough generalization of the argument, covering unitary or non-unitary measurements as well as sharp and unsharp object and pointer observables.

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